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Self-organized critical state in a directed sandpile automaton on Bethe lattices: equivalence to site percolation

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Abstract. We consider a directed sandpile automaton on the Bethe lattice and show that the system evolves ergodically to every configuration with equal probability, irrespective of the initial state. We make it clear that the system is driven into the self-organized critical state which is in equivalence to site percolation at the critical percolation threshold on the Bethe lattice.

Recently the concept of self-organized criticality (SOC) proposed by Bak, Tang and Wiesenfeld (BTW) [1, 2] has aroused much interest. The principal conclusion of BTW is that dissipative dynamical systems tend to organize themselves without fine-tuning into a critical state where chain reactions of all sizes in time and space propagate through the system.

One of the simplest models that show SOC is the cellular automaton model of sandpiles. Our knowledge about this model came first from the computer simulations of BTW [1, 2]. More extensive and large-scale simulations were made later by Kadanoff *et al* [3] and Grassberger and Manna [4, 5]. By the efforts of Dhar and his co-workers [6–8], it seems that it is possible to tackle the problem analytically. A directed version of the sandpile model has been successfully solved in [6]. Using some assumptions about the compactness of avalanche clusters, Zhang [9] has determined the critical exponents characterizing the SOC state in the undirected BTW model in all dimensions. Based on the analogy between the avalanche process and the branching self-avoiding walk and using the ε-expansion technique, Obukhov [10] has conjectured that the upper critical dimension for the undirected BTW model is $d_c = 4$, while $d_c = 3$ for the directed model of reference [6]. For dimensions above d_c, the mean-field values of the critical exponents are valid. It is found that these values are *numerically* identical with the mean-field values of percolation models. The relationship between the sandpile model and the percolation model has been noted previously in [4, 8].

In this paper we devise a simple sandpile automaton which has the advantage of easy solvability and explicit demonstration of the equivalence of the SOC state to the site percolation at the critical percolation threshold in this particular case. Our model is a directed sandpile automaton on the Bethe lattice. The toppling direction is the outgoing direction of the Bethe lattice. Here we consider one branch of the lattice (see figure 1). Each lattice site (including the boundary sites) has a coordination number of $z = m + 1$. The system consists of N sites and the toppling rule is height-dependent. For each site, a sand height h_i is assigned ($0 \leqslant h_i \leqslant m - 1$). The dynamics is defined as follows:

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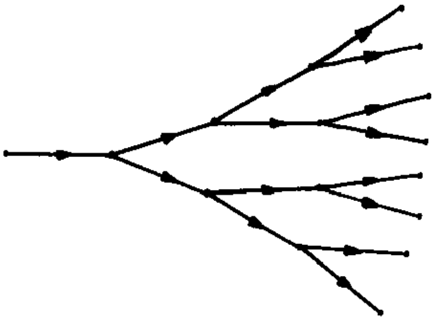

Figure 1. One branch of the Bethe lattice with coordination number $z = m + 1 = 3$. The arrows represent the toppling directions.

(1) when the configuration is stable, a site i is chosen randomly and the sand height h_i is increased by a unit, i.e. $h_i \to h_i + 1$;

(2) if at any site the sand height is greater than $m-1$, its height decreases by m and sand particles drop along the preferred directions (toppling directions) to nearest neighbours (NN). As a result, the height of each of the NN sites in the preferred directions increases by 1. Sand grains drop out of the system from the boundary sites.

We consider the dynamic process starting from a stable configuration. In the phase-space, a configuration is represented by a point, and all the stable configurations form an N-dimensional hypercubic lattice with total point number of m^N. If Prob($\boldsymbol{P}$, t) denotes the probability of the system being found in point $\boldsymbol{P}$ in the phase-space at time t, then the initial state is characterized by:

$$\text{Prob}(\boldsymbol{P}, 0) = \delta_{\boldsymbol{P}\boldsymbol{P}_0} \tag{1a}$$

where $\boldsymbol{P}_0$ represents the initial configuration.

There are N choices of adding a sand grain, with probability of $1/N$ for each, to a stable configuration (say, $\boldsymbol{P}$ in the phase-space), resulting in N new stable configurations (say, $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots, \boldsymbol{Q}_N$). Correspondingly, the probability Prob($\boldsymbol{P}$, t) splits into N equal parts which contribute to Prob($\boldsymbol{Q}_i$, $t+1$), $i = 1, 2, \ldots, N$. Since the topplings are directed and the dynamics is invertible, there are N (and only N) configurations (say, $\boldsymbol{O}_1, \boldsymbol{O}_2, \ldots, \boldsymbol{O}_N$) at time $t-1$ which have the probability of resulting in the configuration $\boldsymbol{P}$ at time t. Therefore,

$$\text{Prob}(\boldsymbol{P}, t) = \frac{1}{N} \sum_{i=1}^{N} \text{Prob}(\boldsymbol{O}_i, t-1). \tag{1b}$$

Equation (1b) is the rule of a new automaton in the phase-space. After a sufficiently long time of applying such a rule, the quantity Prob($\boldsymbol{P}$, t) will become stationary and uniformly distributed, irrespective of the initial condition (1a) (at this stage, the SOC state is reached):

$$\text{Prob}(\boldsymbol{P}, t) = \text{Prob} = \frac{1}{m^N} \qquad \text{for } t \gg 1. \tag{2}$$

In the computer simulations, adding one sand grain to a stable configuration will result in a new configuration, i.e. adding a particle will cause the system to transfer from one point to another in the phase-space. After a sufficiently long time of adding sand grains, the system will be ergodic. Since the probability of the configurations (or ensembles) is uniformly distributed in the SOC state, we can accept that the time average of some quantity (e.g. avalanche size) is equal to its ensemble average. This puts us in a position that we can compare the results of computer simulations (time average) with analytical results (ensemble average).

In order to clarify the relationship between the above sandpile model and the percolation model, it is useful to regard sites with $h_i = m-1$ as 'active' sites and others as 'inactive' sites. Site percolation only involves configuration whose elements are 'occupied' and 'unoccupied', corresponding to 'active' and 'inactive' in the sandpile model. One can see that different configurations in the sandpile model may be the same configuration from the percolation's point of view. A configuration with s sites occupied and $N-s$ sites unoccupied in the percolation model corresponds to $(m-1)^{N-s}$ configurations in the above sandpile automaton. Therefore, for the sandpile model studied, the probability of the system being in a configuration with s 'active' and $N-s$ 'inactive' sites is

$$P(s)=(m-1)^{N-s}\times \mathrm{Prob}=\left(1-\frac{1}{m}\right)^{N-s}\frac{1}{m^s}=p_c^s(1-p_c)^{N-s} \tag{3}$$

with

$$p_c=\frac{1}{m}=\frac{1}{z-1}. \tag{4}$$

The probability expressed by equation (3) has the same form as the probability of the configuration with s site occupied (with probability p_c) and the rest unoccupied in the site percolation model [13]. This indicates that the SOC state of the sandpile automaton studied is identical to the critical state of the percolation model at threshold. Adding sand grains to the system only means ensemble-averaging in the critical state of the percolation model. The critical exponents characterizing the SOC state of the automaton should have the same values as those of the percolation in the Bethe lattice, i.e. the mean-field values of percolation models.

In the following we would like to calculate the average sand height per site. If f_j ($j=0, 1, \ldots, m-1$) denotes the probability that a site has height $h_i = j$, then in the SOC state, the probability for a chosen site flipping from $h_i=0$ to $h_i=1$ should equal the probability of flipping from $h_i=1$ to $h_i=2$. The former is

$$\Pr(0\to 1)=\frac{1}{N}f_0+\frac{1}{N}f_0f_{m-1}+\frac{1}{N}f_0f_{m-1}f_{m-1}+\ldots. \tag{5}$$

On the right-hand side of equation (5), the first term accounts for the possibility of adding a sand grain just to the chosen site, and the other terms are due to the toppling of other sites which are the precedent sites of the chosen one (There are no such simple terms either in the undirected BTW models or in the directed version of [6].)

Similarly,

$$\Pr(1\to 2)=\frac{1}{N}f_1+\frac{1}{N}f_1f_{m-1}+\frac{1}{N}f_1f_{m-1}f_{m-1}+\ldots. \tag{6}$$

Setting equation (5) equal to equation (6) yields

$$f_0=f_1. \tag{7}$$

In a similar manner we can obtain

$$f_0 = f_1 = f_2 = \ldots = f_{m-1} = \frac{1}{m}. \tag{8}$$

Then the average height is

$$\langle h \rangle = \sum_{i=0}^{m-1} if_i = (m-1)/2 = \frac{z}{2} - 1 \tag{9}$$

where z is the coordination number of the Bethe lattice. So we have calculated the average height using equation (8) which is exact here and consistent with the ansatz for the energy distribution of [14] in the large dimension limit [14, equations (4) and (11)].

Finally, note that the directedness in our sandpile automaton is outgoing and this is in accordance with the global flow of sand grains in the undirected BTW model; the result of directedness is no multi-toppling and we know from [4] that multi-topplings are rare and become negligible for large avalanche in undirected BTW models.

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